The limits of y are 0 and $\dfrac{p}{2+\sqrt{2}}=y'$.

$$\therefore \Delta=\frac{\frac{1}{4}\pi\int_0^{y'}\frac{y^2(p-2y)^2}{(p-y)^2}dy}{\int_0^{y'}dy}=\frac{\pi(2+\sqrt{2})}{4p}\int_0^{y'}\left(4y^2+4py+5p^2+\frac{p^4}{(p-y)^2}-\frac{6p^3}{p-y}\right)dy$$

$$=\frac{\pi p^2}{12}[27-4\sqrt{2}-9(2+\sqrt{2})\log 2].$$

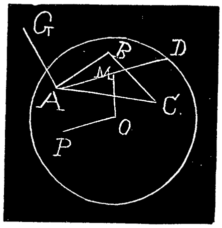
In this solution, as in solution of problem 75, I used the limits of y, 0 and $p/(2+\sqrt{2})$. These limits give all possible variations of size of area. Any other areas are mere repetitions of those included in the above and such a repetition or doubling of areas I believe to be inadmissible.

94. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Three points are taken at random on the surface of the sphere. Find the chance that the triangle thus formed is acute angled.

Solution by the PROPOSER.

Let AD be the diameter of the section of the sphere made by the plane through the three random points A, B, C; M its center; O the center of the sphere; OP a line such that AB is parallel to the plane MOP; p=the chance.



Let $AO=r$, $\angle AOM=\theta$, $\angle GAC=\varphi$, $\angle GAB=\psi$, $\angle MOP=\lambda$, the angle MOP makes with some fixed plane through $OP=\rho$.

An element of the sphere at A is $4\pi r^2\sin\theta d\theta$; at B, $4r^2\sin\theta\sin(\varphi-\psi)\sin\psi\sin\lambda d\psi d\rho$; at C, $4r^2\sin\theta\sin\varphi d\varphi d\lambda$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $\frac{1}{2}\pi$ and π; of ψ, $\pi-\varphi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and 2π.

The three points can be taken $64\pi^3r^6$ ways on the surface of the sphere. Hence

$$p=\frac{1}{64\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{2\pi}\int_0^{\pi}\int_0^{2\pi}4\pi r^2\sin\theta d\theta . 4r^2\sin\theta\sin\varphi d\varphi d\lambda$$

$$\times 4r^2\sin\theta\sin(\varphi-\psi)\sin\psi\sin\lambda d\psi d\rho$$

$$=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\sin^3\theta\sin\varphi\sin\psi\sin(\varphi-\psi)\sin\lambda d\theta d\varphi d\psi d\lambda$$

$$=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{\frac{1}{2}\pi}\sin^3\theta\sin\varphi\sin\psi\sin(\varphi-\psi)d\theta d\varphi d\psi$$

$$=\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\sin^2\theta(4\sin^2\varphi-4\sin^4\varphi+\pi\sin\varphi\cos\varphi-2\varphi\sin\varphi\cos\varphi)d\theta d\varphi$$

$$=\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\sin^3\theta d\theta=\tfrac{1}{3}.$$

MISCELLANEOUS.

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88. **Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.**

Solve to infinity the series $5\cos\theta=\dfrac{7\cos3\theta}{3\,!}+\dfrac{9\cos5\theta}{5\,!}\ldots\ldots$

Solution by the PROPOSER.

Let the series be $C=\dfrac{5\cos\theta}{1}+\dfrac{7\cos3\theta}{3\,!}+\dfrac{9\cos5\theta}{5\,!}+\ldots.$

also, $S=\dfrac{5\sin\theta}{1}+\dfrac{7\sin3\theta}{3\,!}+\dfrac{9\sin5\theta}{5\,!}+\ldots.$

Then $C+Si=\dfrac{5(\cos\theta+i\sin\theta)}{1}+\dfrac{7(\cos3\theta+i\sin3\theta)}{3\,!}+\ldots.$

or using a familiar notation,

$$C+Si=\frac{5e^{i\theta}}{1}+\frac{7\,e^{3i\theta}}{3\,!}+\frac{9\,e^{5i\theta}}{5\,!}+\ldots.$$

which can be written thus:

$$=4\left(e^{i\theta}+\frac{e^{3i\theta}}{3\,!}+\frac{e^{5i\theta}}{5\,!}+\ldots\right)+\left(e^{i\theta}+\frac{3\,e^{3i\theta}}{3\,!}+\frac{5\,e^{5i\theta}}{5\,!}+\ldots\right)$$

$$=4\left(e^{i\theta}+\frac{e^{3i\theta}}{3\,!}+\frac{e^{5i\theta}}{5\,!}+\ldots\right)+e^{i\theta}\left(1+\frac{e^{2i\theta}}{2\,!}+\frac{e^{4i\theta}}{4\,!}+\ldots\right)$$

$$=2\left(e^{e^{i\theta}}-e^{-e^{i\theta}}\right)+\frac{e^{i\theta}}{2}\left(e^{e^{i\theta}}+e^{-e^{i\theta}}\right)$$

But $e^{i\theta}=\cos\theta+i\sin\theta$, so that we have

$$C+Si=2\left(e^{\cos\theta+i\sin\theta}-e^{-\cos\theta-i\sin\theta}\right)+\frac{e^{i\theta}}{2}\left(e^{\cos\theta+i\sin\theta}+e^{-\cos\theta-i\sin\theta}\right)$$

$$=2e^{\cos\theta}\,e^{i\sin\theta}-2e^{-\cos\theta}\,e^{-i\sin\theta}+\frac{e^{\cos\theta}}{2}.e^{i\theta+i\sin\theta}+\frac{e^{-\cos\theta}}{2}\,e^{i\theta-i\sin\theta}$$